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Relaxation of soliton excitation in the collective variable formalism

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Abstract. A variational approach is suggested as a base for constructing the formalized scheme for the method of collective variables. Within this framework the dynamics of the internal modes of soliton excitations is considered. Using a systematic perturbation scheme in the amplitudes of these modes the radiation field is found and the dissipation of soliton excitations is determined.

1. Introduction

The problem of soliton excitations in nonlinear systems has been investigated by the collective variable approach in many papers (for references, see, for example, 1). Generally, the employment of collective variables involves the necessity of eliminating the zero modes that are due to the presence of continuous symmetries in the system. In particular, Tomboulis [2] developed the canonical formalism enabling one to describe the evolution of the Goldstone variable, i.e. the soliton coordinate X_0 of the nonlinear Klein-Gordon equation

$$\ddot{\Phi}(x, t) - \Phi''(x, t) + U'(\Phi(x, t)) = 0 \tag{1}$$

and also of the field fluctuations $\Phi(x, t)$ above the exact soliton solution $\mathcal{U}_c(x - X_0)$. The possibility of using non-Goldstone variables as collective coordinates has been discussed in [3, 4] in connection with the necessity of describing a complicated pattern of soliton scattering in non-integrable systems. In a number of recent publications [5-8], collective variables have been used for describing the internal degrees of freedom of soliton excitations. In such an approach, the soliton is considered as a set of nonlinear oscillators, whose collective coordinates X_i together with the Goldstone variable X_0 are defined in terms of the initial field $\Phi(x)$ by the constraints

$$\left\langle \frac{\delta \Phi_c}{\delta X} (\Phi - \Phi_c) \right\rangle = 0 \qquad \langle A \rangle = \int \mathrm{d}x\, A(x). \tag{2}$$

It is evident that the efficiency of this approach depends on the choice of the ansatz Φ_c. The incorporation of non-Goldstone variables into the number of collective coordinates does not allow one to use the steady-state solutions of (1) as $\Phi_c(x, X)$. This stimulates the use of various artificial models. However, the insufficiency of such an approach is noticeable even in the main approximation. In particular, it does not provide coincidence between the small-amplitude oscillation frequencies of the collective variables and the corresponding frequencies of (1) linearized close to $\mathcal{U}_c(x)$. This fact illustrates the difficulties caused by the absence of an adequate functional relation

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between the collective variables and the initial microscopic ones of the system under consideration. Similar difficulties are peculiar to a number of problems mentioned in section 2. One may say that for the majority of the systems considered such a difficulty is the rule rather than the exception.

In the present paper we suggest a variational approach as a base for constructing a formalized scheme for the method of collective variables. In the framework of this scheme a natural relation between the collective variables and the initial variables is established. This is described in section 2. The rest of the paper is of illustrative character and is concerned with the nonlinear Klein–Gordon equation. In sections 3–5 the transition to new canonical variables is carried out, the equations for the collective (soliton) field components are written in explicit form, and the expression for the Hamiltonian is presented in terms of these new variables. In sections 6 and 7, by means of perturbation theory in internal-mode amplitudes, the ansatz $\Phi_c(x, X)$ is calculated, the radiation field generated by the oscillations of these modes is found and the dissipation of soliton excitations is determined. In section 8 the possibilities of the method for two-soliton state analysis are discussed.

2. Elements of variational formulation for the collective variable method

Here the possibility of formalizing the transition from the initial dynamic variables of the system to the collective variables is considered. Note that the same system may be described by different sets of collective variables. This is connected with both the specific formulation of the problem and its detailed consideration. As soon as the set has been formed on the base of some physical concepts, there arises the necessity of constructing a convenient mathematical scheme. This is caused by the transition to new variables as well as by the need to clarify the structure of the Hamiltonian in terms of these new variables, taking into account the significance of its components. Such a rearrangement is based on adequate division of the initial dynamic variables into collective and radiative (fluctuating) components. To accomplish this division in an acceptable mathematical form note that, in accordance with the main idea of the method, all the remaining variables are subordinate to the collective ones. This means that in the main approximation all other degrees of freedom may be interpreted as 'frozen' into the flow of the collective variables and evolving together with this flow. Considering that the collective variables are similar to the 'external' adiabatic parameters with respect to the initial microscopic variables, it seems reasonable *to define the coherent components of the initial microscopic variables as extremals which minimize the microscopic Hamiltonian of the system at given values of the collective variables.* In other words, if, for example, $\chi_\alpha(Q, P, q, p) = 0$, $\alpha = 1, \ldots 2n$ is the system of constraints which define n pairs of the collective variables Q, P as functions of the initial microscopic variables q^i, p_i, and $H(q, p)$ is the microscopic Hamiltonian, then the coherent components $\bar{q}(Q, P), \bar{p}(Q, P)$ of the initial variables are defined as solutions of the set of equations for the conditional extremum

$$\frac{\delta}{\delta q}(H(q, p) + \lambda_\alpha \chi_\alpha(Q, P, q, p)) = 0$$

$$\frac{\delta}{\delta p}(H(q, p) + \lambda_\alpha \chi_\alpha(Q, P, q, p)) = 0$$

where λ_α are Lagrange multipliers.

The above definition and the equations for extremals play an essential role in the formulation of the collective variable method. Their forms suggest that the trajectory of the system is located within a limited region of the total phase space in the vicinity of the extremals $\bar{q}(Q, P), \bar{p}(Q, P)$. Thus, we may conclude that *the value of the microscopic Hamiltonian $H(q, p)$ corresponding to its minimizing extremals defines the effective Hamiltonian*

$$H_0(Q, P) = H(\bar{q}(Q, P), \bar{p}(Q, P))$$

for the collective variables.

Usually, in macroscopic physics and in field theory this Hamiltonian is a subject of phenomenological research. The subsequent terms of the expansion of the microscopic Hamiltonian $H(q, p)$ near the extremals $\bar{q}(Q, P), \bar{p}(Q, P)$ define the dissipative and radiative properties of the system. These terms are also significant for the determination of low-energy scattering characteristics for collective and non-coherent modes.

The scheme described could have been viewed as complete but for the following essential consideration. The suggested formulation of the collective variable method has implied *a priori* the existence of a definite functional relation between the collective variables Q, P and the microscopic ones q, p. Actually, the macroscopic variables, e.g. the densities of conserved values in hydrodynamic theories, are directly expressed in terms of the initial variables of the system. At the same time, there exist a variety of topical problems characterized by the fact that, *ab initio*, the collective variables which are typical of them have no definite functional relation with the initial dynamic variables. In particular, one may mention the coordinates of extended objects (e.g. domain walls or vortices) in multidimensional theories, soliton coordinates in multisoliton configurations, the parameters of local equilibrium states, i.e. the so-called Goldstone fields.

To formulate a general definition of the relation between the collective coordinates and those of the initial system for the above non-trivial examples, consider the relation for the kink coordinate X_0 of the one-dimensional nonlinear Klein–Gordon equation. In this case, the collective coordinate X_0 as a function of the initial field $\Phi(x)$ is defined from the gauge condition (2) in which the ansatz $\Phi_c(x, X_0)$ is simply the static kink solution $\mathcal{U}_c(x - X_0)$ [2]. In other words, from all possible configurations one chooses only that which minimizes the Hamiltonian of the system $H(\Phi, \Pi)$ corresponding to the kink states with the fixed kink coordinate X_0 as the ansatz $\Phi_c(x, X_0)$.

To generalize the above observations, we define *the dependence of the arbitrary collective coordinates Q ('gauge' coordinates) on the microscopic field coordinates by means of the gauge conditions*

$$\chi_\alpha = \left\langle \frac{\delta q_c(Q)}{\delta Q_\alpha} (q - q_c(Q)) \right\rangle \equiv \frac{\delta q_c^i(Q)}{\delta Q_\alpha} g_{ik}(q_c(Q))(q^k - q_c^k(Q)) = 0$$

where the ansatz $q_c(Q)$ is the quasistatic configuration of the microscopic coordinates. This configuration is defined from its energy minimum condition, when the values of the collective coordinate Q are fixed. Or

$$\frac{\delta}{\delta q}(H(q, 0) + \lambda_\alpha \chi_\alpha(q, Q)) = 0$$

where λ_α are Lagrange multipliers. The brackets $\langle \ldots \rangle$ denote the properly introduced scalar product; $g_{ik}(q)$ is the corresponding metric tensor.

These definitions imply that the initial system is a Lagrange system with the non-degenerate Lagrangian $L(q, \partial q/\partial t)$. Thus, taking into account the fact that in the

presence of the constraints $\chi_\alpha(q, Q) = 0$ the variables transform into the functions of the collective coordinates, we find the momenta which are canonically conjugate to the collective coordinates

$$P_\alpha = \frac{\delta L}{\delta \dot{Q}_\alpha} = \left\langle \frac{\delta q}{\delta Q_\alpha} \frac{\delta L}{\delta \dot{q}} \right\rangle = \left\langle \frac{\delta q}{\delta Q_\alpha} p \right\rangle.$$

Moreover, the definitions of the ansatz $q_c(Q)$ and the extremal $\bar{q}(Q, P)$ yield the relation $q_c(Q) = \bar{q}(Q, 0)$.

Defining the kernel of the projection operator $\mathcal{P}$ by the expression

$$\mathcal{P}^i_k = g_{kl}(q_c) \frac{\delta q^l_c}{\delta Q} M^{(0)-1} \frac{\delta q^i_c}{\delta Q}$$

where

$$M^{(0)}_{\alpha\beta} = \left\langle \frac{\delta q_c}{\delta Q_\alpha} \frac{\delta q_c}{\delta Q_\beta} \right\rangle = \frac{\delta q^i_c}{\delta Q_\alpha} g_{ik}(q_c) \frac{\delta q^k_c}{\delta Q_\beta}$$

we obtain

$$(1 - \mathcal{P}) \frac{\delta H(q, 0)}{\delta q}\bigg|_{q=q_c} = 0$$

from the equation for the ansatz $q_c(Q)$, taking into account the constraints $\chi_\alpha = 0$. Using the expression for $\mathcal{P}^i_k$ let us now present this equation in a more detailed form as

$$\frac{\delta H(q_c, 0)}{\delta q^i_c} = g_{ik}(q_c) \frac{\delta q^k_c}{\delta Q} M^{(0)-1} \frac{\delta E(Q)}{\delta Q} \qquad E(Q) \equiv H(q_c(Q), 0).$$

The self-consistent forces $\delta E/\delta Q$ on the right-hand side of this equation provide an optimum form of the ansatz $q_c(Q)$ for the given values of the collective coordinates Q. For instance, if the coordinates are the degenerate parameters of vacuum solutions, these forces become equal to zero, and the ansatz $q_c(Q)$ coincides with the corresponding vacuum solution of the equation $\delta H/\delta q = 0$.

The equation for the ansatz $q_c(Q)$ contains important information on the character of the interaction of nonlinear excitation and allows one to establish the existence of bound soliton states.

Consideration of the equation for the ansatz $q_c(Q)$ is now complete for the main points of the method. Further details within the framework of the general approach appears unnecessary. All the main elements of the scheme considered arise in the problem discussed below.

3. Canonical transformation

While studying the internal soliton modes of the one-dimensional nonlinear Klein-Gordon system described by the Hamiltonian

$$H = \int \mathrm{d}x [\tfrac{1}{2}\Pi^2(x) + \tfrac{1}{2}\Phi'^2(x) + U(\Phi(x))] \tag{3}$$

we introduce into consideration the collective coordinates X_i corresponding to these modes and the momenta P_i conjugate to X_i. In accordance with the prescriptions of the previous sections, these variables may be expressed as functions of the canonical field pair $\Phi(x), \Pi(x) = \dot{\Phi}(x)$, using the constraints

$$\chi_{i1} \equiv \left\langle \frac{\delta \Phi_c}{\delta X_i} (\Phi - \Phi_c) \right\rangle = 0 \qquad \chi_{i2} \equiv \left\langle \Pi \frac{\delta \Phi}{\delta X_i} \right\rangle - P_i = 0 \tag{4}$$

where $\Phi_c(x, X)$ is a quasistatic ansatz. The equation for this ansatz and the solution of this equation will be given below.

By choosing some basis $\Psi_\lambda(x, X)(\langle \Psi_\lambda \Psi_{\lambda'} \rangle = \delta_{\lambda\lambda'})$ in the subspace of the operator $(1 - \mathcal{P})$, where the kernel of the operator $\mathcal{P}$ has the form

$$\mathcal{P}(x, x') = \frac{\delta \Phi_c(x)}{\delta X_i} M^{(0)-1}_{ik} \frac{\delta \Phi_c(x')}{\delta X_k} \qquad M^{(0)}_{ik} = \left\langle \frac{\delta \Phi_c}{\delta X_i} \frac{\delta \Phi_c}{\delta X_k} \right\rangle \tag{5}$$

in the given model, we may define the explicit dependence of the field variables on X, using the expansions

$$\Phi(x, X) = \Phi_c(x, X) + \sum_\lambda a_\lambda \Psi_\lambda(x, X) \qquad \pi(x, X) = (1 - \mathcal{P})\Pi(x) = \sum_\lambda b_\lambda \Psi_\lambda \tag{6}$$

with the coefficients a_λ, b_λ considered as independent.

Then using the formulae

$$P = \left\langle \Pi \frac{\delta \Phi}{\delta X} \right\rangle = \left\langle \Pi (1 - \mathcal{P}) \frac{\delta \Phi}{\delta X} \right\rangle + \left\langle \Pi \mathcal{P} \frac{\delta \Phi}{\delta X} \right\rangle = G + Q M^{(0)-1} \left\langle \frac{\delta \Phi_c}{\delta X} \Pi \right\rangle$$

where

$$G_i = \left\langle \frac{\delta \Phi}{\delta X_i} \pi \right\rangle \qquad Q_{ik} = \left\langle \frac{\delta \Phi}{\delta X_i} \frac{\delta \Phi_c}{\delta X_k} \right\rangle \qquad Q_{ik} = Q_{ki} \tag{7}$$

and

$$\Pi(x) = (1 - \mathcal{P})\Pi(x) + \mathcal{P}\Pi(x) = \pi(x) + \left\langle \frac{\delta \Phi_c}{\delta X} \Pi \right\rangle M^{(0)-1} \frac{\delta \Phi_c(x, X)}{\delta X}$$

we obtain the expression for the initial field momentum $\Pi(x)$ in terms of the new variables:

$$\Pi(x) = \pi(x) + (P - G) Q^{-1} \frac{\delta \Phi_c(x, X)}{\delta X}. \tag{8}$$

According to (4), (6) and (8) we have $\langle \dot{\Phi} \Pi \rangle = \dot{X} P + \dot{a} b$, and since $\Phi(x), \Pi(x)$ are a canonical pair, it means that the new variables X, P and a, b are canonical too, with the following non-zero Poisson brackets

$$\{X_i, P_k\} = \delta_{ik} \qquad \{a_\lambda, b_{\lambda'}\} = \delta_{\lambda\lambda'}. \tag{9}$$

Note, however, that the variables a, b play an auxiliary role. It is their fluctuations or, more exactly, the fluctuations of the $\Phi(x)$ and $\pi(x)$ fields against the background of their coherent (soliton) components that have a physical sense. Generally speaking, the dynamic variables corresponding to these fluctuations are no longer canonical.

The results adduced in this section are a generalization of the canonical transformation [2] for the case of arbitrary collective variables.

4. Equations for the soliton components $\bar{\Phi}(x, X, P)$ and for the ansatz $\Phi_c(x, X)$

As pointed out in section 2, the collective (soliton) components of the initial fields $\Phi(x), \Pi(x)$ play an essential role in building and using the method of collective variables. These components are defined as extremals which minimize the Hamiltonian $H(\Phi, \Pi)$ for the fixed values of the variables X, P. The ansatz $\Phi_c(x, X)$ is directly connected with these components. By definition, this ansatz is equal to the soliton component $\bar{\Phi}(x, X, P)$ when $P = 0$:

$$\Phi_c(x, X) = \bar{\Phi}(x, X, P)|_{P=0}. \tag{10}$$

Taking into account the constraints (4) by introducing the Lagrange multipliers

$$\mathcal{H} = H + \lambda_1 \chi_1 + \lambda_2 \chi_2$$

and observing that, according to (4) and (6),

$$\delta P(\Phi, \Pi) = \left\langle \delta\Pi \frac{\delta\Phi}{\delta X} \right\rangle - \left\langle \delta\Phi \frac{\delta\Pi}{\delta X} \right\rangle \tag{11}$$

we find the equations defining the conditional extremum

$$\frac{\delta\mathcal{H}}{\delta\Phi} \equiv \frac{\delta H}{\delta\Phi} + \lambda_1 \frac{\delta\Phi_c}{\delta X} - \lambda_2 \frac{\delta\Pi}{\delta X} = 0 \tag{12}$$

$$\frac{\delta\mathcal{H}}{\delta\Pi} \equiv \frac{\delta H}{\delta\Pi} + \lambda_2 \frac{\delta\Phi}{\delta X} = 0. \tag{13}$$

Using (13), let us consider the equations

$$\left\langle \frac{\delta\mathcal{H}}{\delta\Pi} \frac{\delta\Phi_c}{\delta X} \right\rangle = 0 \qquad \left\langle \frac{\delta\mathcal{H}}{\delta\Pi} \frac{\delta\Phi}{\delta X} \right\rangle = 0.$$

From the first of this pair, in view of (8), it follows that the factor $-\lambda_{2i}$ coincides with the velocity $\dot{X}_i$:

$$\lambda_2 = -\left\langle \frac{\delta H}{\delta\Pi} \frac{\delta\Phi_c}{\delta X} \right\rangle Q^{-1} \equiv -\frac{\delta H}{\delta P} = -\dot{X}.$$

The second of the two equations considered gives the value of this velocity at the extremum point:

$$-\lambda_2 = \dot{X} = M^{-1} P \qquad M_{ik} \equiv \left\langle \frac{\delta\Phi}{\delta X_i} \frac{\delta\Phi}{\delta X_k} \right\rangle \Big|_{\Phi=\bar{\Phi}}. \tag{14}$$

By action of the operator $(1 - \mathcal{P})$ on expression (12) we shall now find the equation for the extremal $\bar{\Phi}(x, X, P)$ [9],

$$(1 - \mathcal{P}) \left\{ \frac{\delta H}{\delta\Phi} + P M^{-1} \frac{\delta\Pi}{\delta X} \right\} \Big|_{\substack{\Phi=\bar{\Phi} \\ \pi=\bar{\pi}}} = 0 \tag{15}$$

where, according to (13) and (14), we have

$$\bar{\pi} = (1 - \mathcal{P})\bar{\Pi} \qquad \bar{\Pi} = P M^{-1} \frac{\delta\Phi}{\delta X} \Big|_{\Phi=\bar{\Phi}}. \tag{16}$$

It is evident that by virtue of the definition (10) and (5) and (15), the equation for the ansatz $\Phi_c(x, X)$ has the following form:

$$(1-\mathscr{P})\frac{\delta H(\Phi, 0)}{\delta \Phi}\bigg|_{\Phi=\Phi_c}=0. \tag{17}$$

The solution of (15), which satisfies the kink asymptotics

$$\bar{\Phi}(x, \bar{X}, \bar{P})-\mathscr{U}_c(x-\bar{X}_0)\xrightarrow[x\to\pm\infty]{} 0$$

can be considered as a non-stationary soliton excitation characterized by a set of the collective coordinates X and the momenta P. According to (15) and (17), this solution at small P can be written as

$$\bar{\Phi}(x, X, P)=\left[1-(1-\mathscr{P})L^{-1}(\Phi_c)(1-\mathscr{P})\left(PM^{(0)-1}\frac{\delta}{\delta X}\right)^2\right]\Phi_c(x, X) \tag{18}$$

where

$$L(\Phi_c)\equiv -\frac{\partial^2}{\partial^2 x}+U''(\Phi_c(x, X)).$$

5. Rearrangement of the Hamiltonian

Owing to the division of the initial field into soliton and non-coherent parts, and the general definition of the effective Hamiltonian $H_0(Q, P)$ for the collective variables given in section 2, the total Hamiltonian considered as a functional of the fluctuations $\tilde{\Phi}=\Phi-\bar{\Phi}$, $\tilde{\pi}=\pi-\bar{\pi}$ may be presented as

$$H=H_0(X, P)+H_2(\tilde{\Phi}, \tilde{\pi}, X, P)+V(\tilde{\Phi}, X, P)$$

where, according to (3), (14) and (16),

$$H_0(X, P)=\tfrac{1}{2}PM^{-1}P+H(\bar{\Phi}, 0)$$

is the Hamiltonian defining the elastic dynamics of the collective variables. Since $\mathscr{P}\tilde{\Phi}=0$, and by virtue of (17) and (18),

$$\bar{\Phi}(x, X, P)=\Phi_c(x, X)+O(P^2) \qquad \frac{\delta H}{\delta \Phi}\bigg|_{\Phi=\Phi_c}\tilde{\Phi}=0$$

then to an accuracy of the order of P^2, the Hamiltonian $\bar{H}_0$ can be written as

$$H_0(X, P)=\tfrac{1}{2}PM^{(0)-1}P+H(\Phi_c, 0)+O(P^4). \tag{19}$$

By virtue of (15) and (16) and (11), the linear term in the expansion of the Hamiltonian in the fluctuations $\tilde{\Phi}$, $\tilde{\pi}$ is absent. The exact expression for the squared term H_2 is rather cumbersome. If we observe that, according to (7) and (16),

$$\tilde{\pi}\sim P^3 \qquad \bar{G}\equiv\left\langle\frac{\delta\Phi}{\delta X}\tilde{\pi}\right\rangle\bigg|_{\substack{\Phi=\bar{\Phi}\\ \pi=\bar{\pi}}}\sim P^5$$

$${}^1G\equiv\left\langle\frac{\delta\tilde{\Phi}}{\delta X}\bar{\pi}\right\rangle+\left\langle\frac{\delta\Phi}{\delta X}\bigg|_{\Phi=\bar{\Phi}}\tilde{\pi}\right\rangle\sim P^2$$

we may write H_2 to an accuracy of the order of P:

$$H_2(\tilde{\Phi}, \tilde{\pi}, X, P) = \tfrac{1}{2}\left(\langle \tilde{\Phi} L(\Phi_c)\tilde{\Phi}\rangle + \langle \tilde{\pi}^2\rangle - 2PM^{(0)-1}\left\langle \frac{\delta\tilde{\Phi}}{\delta X}\,\tilde{\pi}\right\rangle\right). \tag{20}$$

The expression for $V(\tilde{\Phi}, X, P)$ has the form

$$V(\tilde{\Phi}, X, P) = \sum_{n=3}^{\infty} \frac{1}{n!}\langle U^{(n)}(\bar{\Phi})\tilde{\Phi}^n\rangle.$$

The variables $\tilde{\Phi}$, $\tilde{\pi}$ define the field fluctuations above the oscillating soliton excitation $\bar{\Phi}$, $\bar{\pi}$ and, therefore, are not canonical. However, at small $\tilde{\Phi}$, $\tilde{\pi}$, P their Poisson brackets are rather similar, according to (6), (9) and (16), to the Poisson brackets that are characteristic of the canonical variables:

$$\begin{aligned} &\{\tilde{\Phi}(x), \tilde{\Phi}(x')\} \sim O(P^3) \qquad \{\tilde{\pi}(x), \tilde{\pi}(x')\} \sim O(P^5) \\ &\{\tilde{\Phi}(x), \tilde{\pi}(x')\} = (1-\mathscr{P})(x, x') + O(P^4). \end{aligned} \tag{21}$$

In conclusion, it should be stressed that the approximation of the H_0 and H_2 fragments by (19) and (20) is suitable for describing 'slow' processes such as low-amplitude oscillations of soliton excitations, low-temperature response to a weak external perturbation [10], etc.

6. Solution of the equation for the ansatz $\Phi_c(x, X)$

Considering the excited states of a soliton as a system of nonlinear oscillators, it is natural to seek the solution of the (17) in the form of a series:

$$\Phi_c(x, X) = \mathscr{U}_c(x - X_0) + \sum_{i=1}^{N} \sigma_i(x - X_0)X_i + \tfrac{1}{2}\sum_{i,k=1}^{N} \sigma_i(x - X_0)X_iX_k + \dots. \tag{22}$$

In the zeroth and first orders in X_i, (17) leads to the corresponding equations for $\mathscr{U}_c$ and σ_i:

$$\left.\frac{\delta H}{\delta \Phi}\right|_{\Phi=\mathscr{U}_c} = 0 \tag{23}$$

$$(1 - \mathscr{P}^{(0)})L\sigma_i = 0 \qquad L = -\left(\frac{\partial}{\partial x}\right)^2 + U''(\mathscr{U}_c) \tag{24}$$

where $\mathscr{P}^{(0)}$ is the projection operator constructed from the vector $\mathscr{U}_c'$, σ_i. The solution of (23) is the static soliton, while (24) is satisfied only by the eigenfunctions of the operator L (but not by their superposition). This becomes evident when it is considered that, according to the definition of $\mathscr{P}^{(0)}$, we have $(1 - \mathscr{P}^{(0)})\sigma_i = 0$. As solutions of (24) that satisfy the zero boundary conditions at $x \to \pm\infty$ we take the eigenfunctions of the discrete spectrum of the operator L with non-zero eigenvalues,

$$L\sigma_i = \omega_i^2\sigma_i \qquad \omega_i \neq 0 \qquad i = 1, \dots, N. \tag{25}$$

By virtue of the invariance of (17) under arbitrary non-degenerate transformations of the collective variables $X_i \to X_i' = X_i'(X_k)$, $i, k = 0, 1, \dots, N$ all higher terms of expansion (22) can be considered orthogonal to the subspace of the discrete spectrum of the operator L:

$$\mathscr{P}^{(0)}\sigma_{i_1,\dots ik} = 0 \qquad k > 1. \tag{26}$$

The arbitrariness under scale transformations of the internal modes $X_i \to X_i' = \alpha_i X_i$, $i = 1, \ldots, N$ can be eliminated by putting $m_{ii} \equiv \langle \sigma_i^2 \rangle = 1$. In this case, the Hamiltonian $H_0(X, P)$ (19), corresponding to the solution for $\Phi_c(x, X)$, will have the following form:

$$H_0 = \frac{P_0^2}{2m} + \tfrac{1}{2} \sum_{i=1}^{N} (P_i^2 + \omega_i^2 X_i^2) \qquad m = \langle \mathcal{U}_c'^2 \rangle. \tag{27}$$

Hence we have the coincidence of the small-oscillation frequencies of the collective variables X with the eigenfrequencies of the operator L, which was mentioned in the introduction.

As will be shown later, the radiation field value can be obtained by calculating the ansatz $\Phi_c(x, X)$ to an accuracy of the terms squared in the collective variables. According to (17), (22) and (25), the equation for the coefficients σ_{ik} has the form

$$(1 - \mathcal{P}^{(0)})[(L - \omega_i^2 - \omega_k^2)\sigma_{ik} + U'''(\mathcal{U}_c)\sigma_i\sigma_k] = 0 \tag{28}$$

which is consistent with the requirement (26). The kernel of the operator of projection on to the subspace of the continuous spectrum of the operator L can be written as

$$(1 - \mathcal{P}^{(0)})(x, y) = \frac{1}{2\pi} \int \mathrm{d}q\, a^{-1}(q)\varphi(x, q)\Psi(y, q)$$

$$= \frac{1}{2\pi} \int \mathrm{d}q\, a^{-1}(q)\Psi(x, q)\varphi(y, q)$$

where $\varphi(x, q), \Psi(x, q)$ are the Jost functions of the operator $L - \Omega^2$,

$$L\varphi(q) = (\Omega^2 + q^2)\varphi(q) \qquad L\Psi(q) = (\Omega^2 + q^2)\Psi(q)$$

which are defined by the asymptotics

$$\varphi(x, q) \to \mathrm{e}^{-\mathrm{i}qx} \qquad x \to -\infty$$

$$\Psi(x, q) \to \mathrm{e}^{\mathrm{i}qx} \qquad x \to \infty$$

$a^{-1}(q)$ is the forward scattering amplitude and Ω^2 is the boundary of the continuous spectrum of the operator L. It is known that the function $a^{-1}(q)$ is analytical in the upper half-plane, with the exception of the discrete spectrum points $q_l = \mathrm{i}\sqrt{\Omega^2 - \omega_l^2}$, $l = 0, 1, \ldots, N$; $\omega_0 = 0$, where it has simple poles. Also, in the upper half-plane of the variable q, the functions $\mathrm{e}^{\mathrm{i}qx}\varphi(x, q)$, $\mathrm{e}^{-\mathrm{i}qx}\Psi(x, q)$ are analytical at any x. Therefore, the solution of (28) satisfying the zero boundary conditions at $x \to \pm\infty$ can be written as

$$\sigma_{ik}(x) = -\frac{1}{2\pi} \int_c \mathrm{d}q\, a^{-1}(q)(L - \omega_i^2 - \omega_k^2)^{-1}\varphi(x, q)\langle \Psi(q)\, U'''(\mathcal{U}_c)\sigma_i\sigma_k \rangle. \tag{29}$$

The position of the integration contour depends on the sign of $\omega_i^2 + \omega_k^2 - \Omega^2$. If $\omega_i^2 + \omega_k^2 < \Omega^2$, then the operator $L - \omega_i^2 - \omega_k^2$ has no zero modes and the solution of (28) is determined by (29), where the integration is performed along the real axis. If $\omega_i^2 + \omega_k^2 > \Omega^2$, the integration is performed along the contour shown in figure 1. In this case, the residues with respect to the points $q = \pm\sqrt{\omega_i^2 + \omega_k^2 - \Omega^2}$ are the zero modes of the operator $L - \omega_i^2 - \omega_k^2$, and the integral in the sense of the principal value is the partial solution of the inhomogeneous equation (28). The oscillating asymptotics of each of these contributions are mutually compensated under integration along the contour depicted in the figure. Knowing σ_{ik}, it is possible to find the field $\bar{\Phi}$ in the main

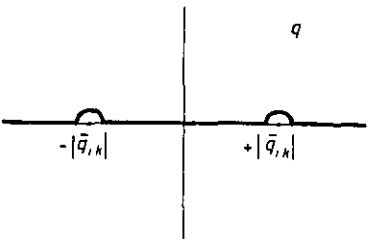

Figure 1. The contour integration in the plane of the complex variable q (equation (29)) providing the zeroth limiting conditions for $\sigma_{ik}(x)$ at $x \to \pm\infty$ in the case $\tilde{q}_{ik}^2 \equiv \omega_i^2 + \omega_k^2 - \Omega^2 > 0$.

approximation. With the Goldstone mode momentum equal to zero, $P_0 = 0$, it follows from (18) and (22) that

$$\bar{\Phi}_c(x, X, P) = \mathcal{U}_c(x - X_0) + \sum_{i=1}^{N} \sigma_i(x - X_0)X_i + \tfrac{1}{2} \sum_{i,k=1}^{N} (X_iX_k - 2P_iP_kL^{-1})\sigma_{ik}(x - X_0). \tag{30}$$

7. Equations for field fluctuations

Internal mode oscillations of soliton excitation generate fluctuations of the variables $\tilde{\Phi}$, $\tilde{\pi}$ and, hence, lead to the radiation of continuous spectrum modes. We shall assume the oscillation amplitudes X to be small and, hence, their conjugate momenta P and the fluctuations of the field variables $\tilde{\Phi}$, $\tilde{\pi}$ will also be small. We now construct the equations for the radiation field, calculating the coefficients of zero and first powers of the fluctuations $\tilde{\Phi}$, $\tilde{\pi}$ to the main approximation in X, P. Neglecting the Goldstone mode X_0 fluctuations in the approximation considered and putting $P_0 = 0$, from (27) and (30) we find

$$\{\tilde{\Phi}, H_0\} = \frac{\delta\tilde{\Phi}}{\delta X}\frac{\delta H_0}{\delta P} - \frac{\delta\tilde{\Phi}}{\delta P}\frac{\delta H_0}{\delta X} \approx \frac{\delta\bar{\Phi}}{\delta P}\frac{\delta H_0}{\delta X} = -2\sum_{i,k} \omega_i^2 P_k X_i L^{-1}\sigma_{ik}$$

$$\{\tilde{\pi}, H_0\} \sim P^4, X^2P^2.$$

Also, according to (20) and (21), we have

$$\{\tilde{\Phi}, H_2\} \approx \tilde{\pi} \qquad \{\tilde{\pi}, H_2\} \approx -L\tilde{\Phi}.$$

So, the set of equations for the field fluctuations will have the form

$$\dot{\tilde{\Phi}} = \tilde{\pi} - 2\sum_{i,k} \omega_i^2 P_k X_i L^{-1}\sigma_{ik} \qquad \dot{\tilde{\pi}} = -L\tilde{\Phi} \tag{31}$$

or

$$\ddot{\tilde{\Phi}} + L\tilde{\Phi} = -2\sum_{i,k} \omega_i^2 L^{-1}\sigma_{ik}\frac{d}{dt} P_k X_i \equiv I(t) \tag{32}$$

where in the zero approximation

$$X_i(t) = \tfrac{1}{2}[A_i \exp(i\omega_i t) + cc] \qquad P_i(t) = \frac{i}{2}\omega_i[A_i \exp(i\omega_i t) - cc].$$

The field $\tilde{\Phi}$ generated by the source $I(t)$ can be written as

$$\tilde{\Phi}(t)=\int_{-\infty}^{t} G_{\mathrm{ret}}(t-t')I(t')\,\mathrm{d}t'$$

where

$$G_{\mathrm{ret}}(t)=\frac{\sin\sqrt{L}t}{\sqrt{L}}\theta(t)$$

is the retarded Green function of (32). Integrating with respect to t' and using the solution (29) for σ_{ik}, we obtain

$$\begin{aligned}\tilde{\Phi}(x,t)=-\frac{1}{4\pi}\sum_{i,k}\omega_i^2\omega_k\int_{\mathrm{c}}\mathrm{d}q\,a^{-1}(q)\{&A_iA_k\exp(\mathrm{i}\omega_{ik}t)\omega_{ik}(L-\omega_{ik}^2+\mathrm{i}\varepsilon)^{-1}\\&+A_i^*A_k\exp[\mathrm{i}(\omega_k-\omega_i)t](\omega_k-\omega_i)[L-(\omega_k-\omega_i)^2]^{-1}\}L^{-1}(L-\omega_i^2-\omega_k^2)^{-1}\\&\times\varphi(x,q)\langle\Psi(q)U'''(\mathcal{U}_{\mathrm{c}})\sigma_i\sigma_k\rangle+\mathrm{cc}\qquad\varepsilon\to+0\qquad\omega_{ik}\equiv\omega_i+\omega_k.\end{aligned}\tag{33}$$

As mentioned above, the integration contour is either coincident with the real axis (at $\omega_i^2+\omega_k^2<\Omega^2$) or turns round the singularities at points $q=\pm\sqrt{\omega_i^2+\omega_k^2-\Omega^2}$ in the upper half-plane. This means that the finite contribution to the integral (33) at $x\to\pm\infty$ arises only from the zeros of the denominator $L-\omega_{ik}^2\pm\mathrm{i}\varepsilon$ lying in the upper half-plane and tending to the real axis at points $q=\pm q_{ik}=\pm\sqrt{\omega_{ik}^2-\Omega^2}$ at $\varepsilon\to+0$. Thus, the asymptotic behaviour of the field $\tilde{\Phi}$ of diverging waves will have the form

$$\tilde{\Phi}(x,t)\underset{x\to\mp\infty}{=}\pm\frac{\mathrm{i}}{16}\sum\frac{A_i^*A_k^*}{q_{ik}a(q_{ik})}\langle\Psi(q_{ik})U'''(\mathcal{U}_{\mathrm{c}})\sigma_i\sigma_k\rangle\exp[-\mathrm{i}(\omega_{ik}t\pm q_{ik}x)]\theta(\omega_{ik}-\Omega)+\mathrm{cc}\tag{34}$$

where we took into account that, for the even scattering potentials, $\varphi(-x,q)=\Psi(x,q)$. So, to second order in the oscillator amplitudes, only waves with frequencies $\omega_{ik}>\Omega$ may radiate.

The energy flux associated with the radiation field (34) is given by $Q(x)=-\dot{\Phi}(x)\tilde{\Phi}'(x)$. If there is only one internal mode with frequency $\omega_1>\Omega/2$, then the energy flux averages over the oscillation period will be

$$\bar{Q}(\mp\infty)=\mp\frac{|\langle\Psi(q_1)U'''(\mathcal{U}_{\mathrm{c}})\sigma_1^2\rangle|^2E^2}{16|a(q_1)|^2q_1\omega_1^3}\qquad q_1=\sqrt{4\omega_1^2-\Omega^2}$$

where $E=\frac{1}{2}|A|^2\omega_1^2$ is the oscillator energy. The oscillator energy dissipation under weak radiation is given by the equation $\dot{E}=-2|\bar{Q}|$, the solution of which is

$$E^{-1}(t)-E^{-1}(0)=\frac{|\langle\Psi(q_1)U'''(\mathcal{U}_{\mathrm{c}})\sigma_1^2\rangle|^2}{16|a(q_1)|^2q_1\omega_1^3}t.$$

To exemplify, we may consider the soliton excitations in the model Φ^4, the frequency internal mode of which satisfies the inequality $\omega_1>\Omega/2$. For this model the potential $U(\Phi)$, the kink solutions $\mathcal{U}_{\mathrm{c}}(x)$ and the eigenfunctions of the discrete and continuous

spectra have the forms

$$U(\Phi)=\frac{\lambda}{4}\left(\Phi^2-\frac{m^2}{\lambda}\right)^2 \qquad \mathcal{U}_c(x)=\frac{m}{\sqrt{\lambda}}\tanh z \qquad z=\frac{mx}{\sqrt{2}}$$

$$\sigma_1(x)=\left(\frac{3m}{2\sqrt{2}}\right)^{1/2}\frac{\sinh z}{\cosh^2 z} \qquad \omega_1^2=\tfrac{3}{2}m^2$$

$$\varphi(x,q)=e^{-iqx}\frac{3\tanh^2 z-1-(\sqrt{2}q/m)^2+(3i\sqrt{2}q/m)\tanh z}{2-(\sqrt{2}q/m)^2-3i\sqrt{2}q/m} \qquad \omega_q^2=q^2+2m^2.$$

The reflection coefficient is, in this case, equal to zero and a trivial integration gives

$$E^{-1}(t)-E^{-1}(0)=\frac{9\sqrt{6}\,\pi^2\lambda t}{20\sinh^2(\pi)m^2}.$$

At $\omega_1<\Omega/2$, the approximation considered gives no radiation and, hence, there is no energy dissipation. In this case, in (31) one must take into account the higher terms in the oscillator amplitudes. In particular, if $\Omega/3<\omega_1<\Omega/2$, then the source in (33) must be calculated taking account of terms of third order in amplitude. It is evident that the energy in this case will be $t^{-1/2}$. If $\omega\ll\Omega$, the dissipation is very low, and calculation of the radiation intensity becomes technically a difficult problem.

8. Conclusion

In the present paper we have formulated the variation principle which allows one to describe the dynamics of various nonlinear excitations in a universal form. The analytical possibilities of this method are connected with the possibilities of the solution of the equation for the ansatz $q_c(Q)$ which defines the quasistatic configuration theory of these excitations. The possibilities mentioned seem to be restricted by the framework of perturbation theory. Depending on the subject under consideration, expansion parameters can be applied to, for example, the curvature and thickness of multidimensional walls, vortices or, as in the case considered above, internal mode amplitudes. The use of the given scheme and, primarily, of the equation for the ansatz $q_c(Q)$, is also of interest from the point of view of two-soliton configuration investigations. For instance, in the nonlinear Klein-Gordon system the ansatz $\Phi_c(x, R)$, which describes the quasistatic two-soliton (soliton–antisoliton) configuration in the centre-of-mass system, satisfies the equation [9]

$$-\frac{\partial^2}{\partial x^2}\Phi_c(x,R)+U'(\Phi_c(x,R))=\frac{\partial\Phi_c(x,R)}{\partial R}M^{(0)-1}\frac{\partial E(R)}{\partial R} \tag{35}$$

where R is the distance between solitons, $E(R)=H(\Phi_c(R),0)$ is the energy of the two-soliton configuration and $M^{(0)}=\int dx(\partial\Phi_c(x,R)/\partial R)^2$. The advantage of this equation is that information on the structure of the one-soliton solution and, consequently, on the structure of the pair when R is large allows one to numerically reconstruct the two-soliton configuration $\Phi_c(x, R)$ and the two-soliton potential $E(R)$ for any R. A similar analysis carried out for analogous equations (or sets of equations) in other models allows us to build the two-soliton potential as a function of R and other collective coordinates, which define the mutual pair orientation in the internal space

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